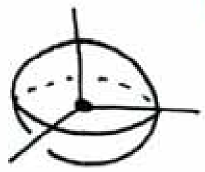
How Einstein "saw" spheres.

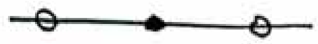
相対性理論とメービウス幾何学のつながり

A sphere in $\mathbb{R}^3$:



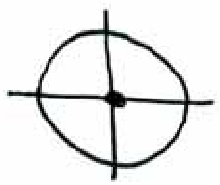
$$= \{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_1^2 + x_2^2 + x_3^2 = 1\},$$

a "sphere" in $\mathbb{R}^1$:



$$= \{\pm 1\},$$

a "sphere" in $\mathbb{R}^2$:



$$= \{(x_1, x_2) \in \mathbb{R}^2 \mid x_1^2 + x_2^2 = 1\},$$

a "sphere" in $\mathbb{R}^n$:

$$\{(x_1, \cdots, x_n) \in \mathbb{R}^n \mid x_1^2 + x_2^2 + \cdots + x_n^2 = 1\}.$$

Here, "distance" is

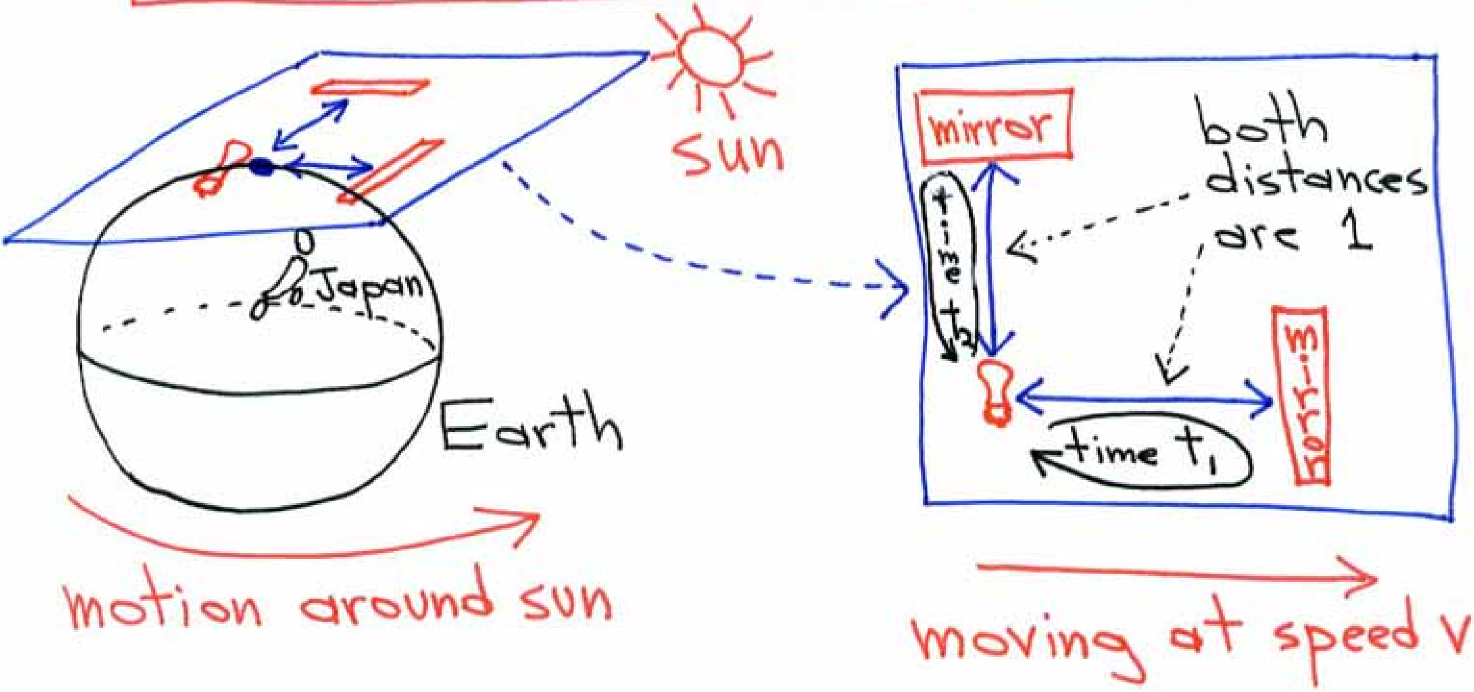
$$\|(x_1,\cdots,x_n)\|^2 = \langle (x_1,\cdots,x_n),(x_1,\cdots,x_n)\rangle = x_1^2+\cdots+x_n^2,$$

where, with $\vec{X}=(x_1,\cdots,x_n)$ and $\vec{Y}=(y_1,\cdots,y_n)$,

$$\langle \vec{X},\vec{Y}\rangle = x_1y_1+\cdots+x_ny_n = \cos(\measuredangle(\vec{X},\vec{Y}))\cdot\|\vec{X}\|\cdot\|\vec{Y}\|.$$

But sometimes we want to use different notions of distance, like in Einstein's relativity, for example. Consider this:

$\boxed{\text{Michelson - Morley experiment}}$

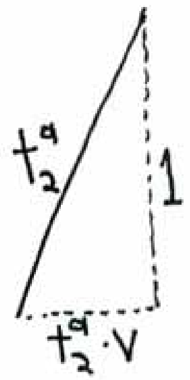


Take speed of light to be 1.

What would happen with respect to Newtonian physics?

$$t_1 = t_1^{a\,(\text{outgoing})} + t_1^{b\,(\text{return})}$$

$$t_1^a = 1 + t_1^a v \;,\quad t_1^b = 1 - t_1^b v \;, \quad \text{so}$$



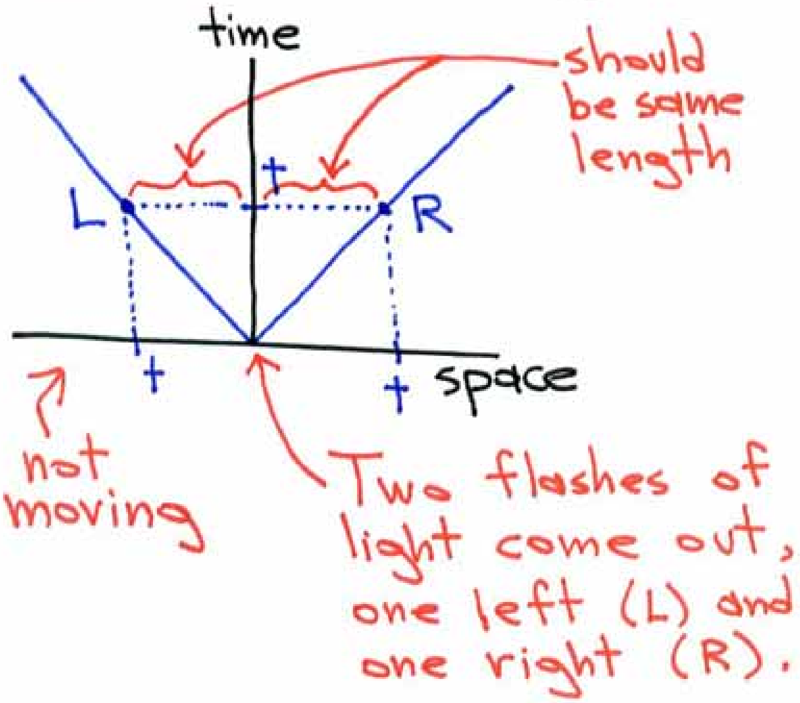
$$t_1 = \frac{1}{1-v} + \frac{1}{1+v} = \frac{2}{1-v^2} \,.$$

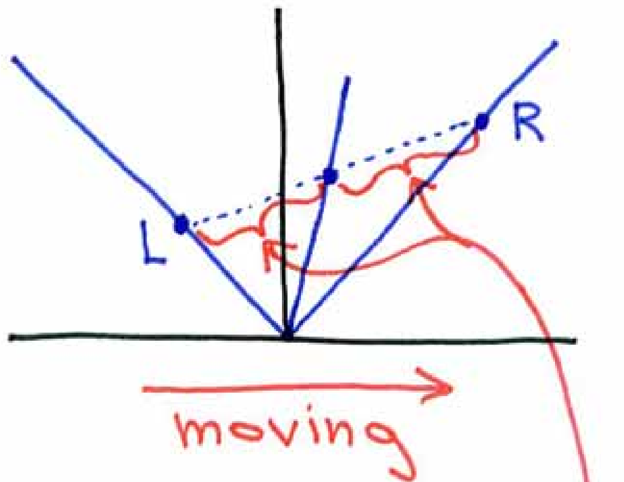
$$t_2 = t_2^a + t_2^b = 2 \cdot t_2^a = \frac{2}{\sqrt{1-v^2}} \,. \quad \text{So}$$

$$t_1 \neq t_2 \,.$$

But the actual experiment gave $t_1 = t_2$.

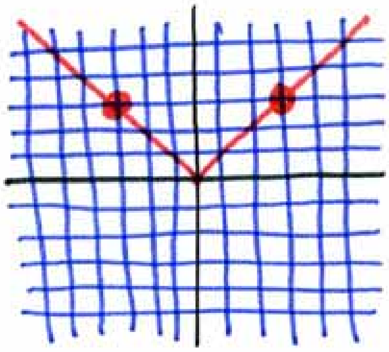
Einstein thought of it this way:

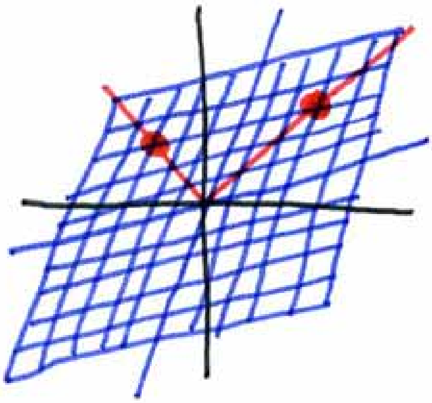




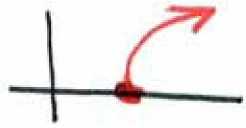
Einstein assumed speed of light <u>always</u> 1, so these two lengths should be the same. So the line from L to R cannot be horizontal.

Now the events L and R do <u>not</u> happen at the same time for the non-moving observer. This solved the Michelson-Morley paradox.





So the motion is

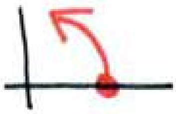


$(x,y) \to (x\cdot\cosh\theta + y\cdot\sinh\theta,\ x\cdot\sinh\theta + y\cdot\cosh\theta)$

$\left\{ \cosh\theta = \dfrac{e^{\theta}+e^{-\theta}}{2},\ \sinh\theta = \dfrac{e^{\theta}-e^{-\theta}}{2} \right\}$

and $x^2 - y^2$ is preserved,

unlike the rotation



$(x,y) \to (x\cdot\cos\theta - y\cdot\sin\theta,\ x\cdot\sin\theta + y\cdot\cos\theta)$,

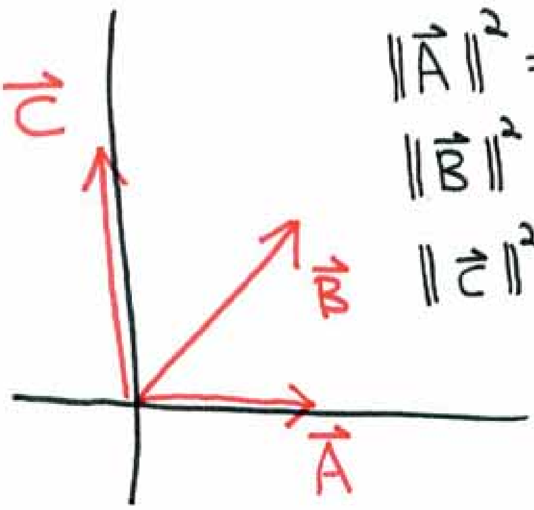
where $x^2 + y^2$ is preserved, in the usual $\mathbb{R}^2$.

So, in relativity, we define distance by

$$x^2 - y^2,$$

or more generally

$$x_1^2 + x_2^2 + \cdots + x_{n-1}^2 - x_n^2 .$$

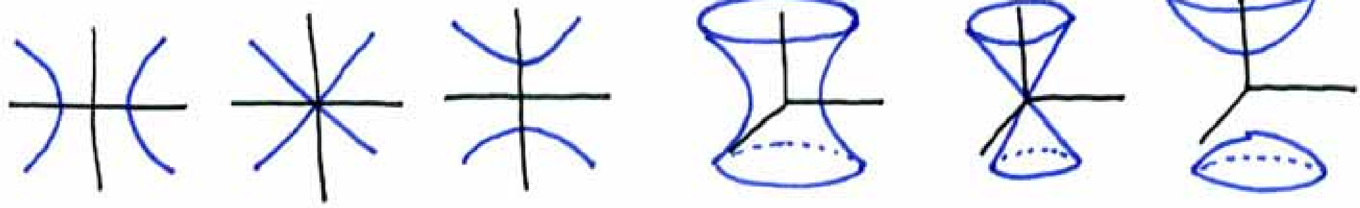


$\|\vec{A}\|^2 = +1$

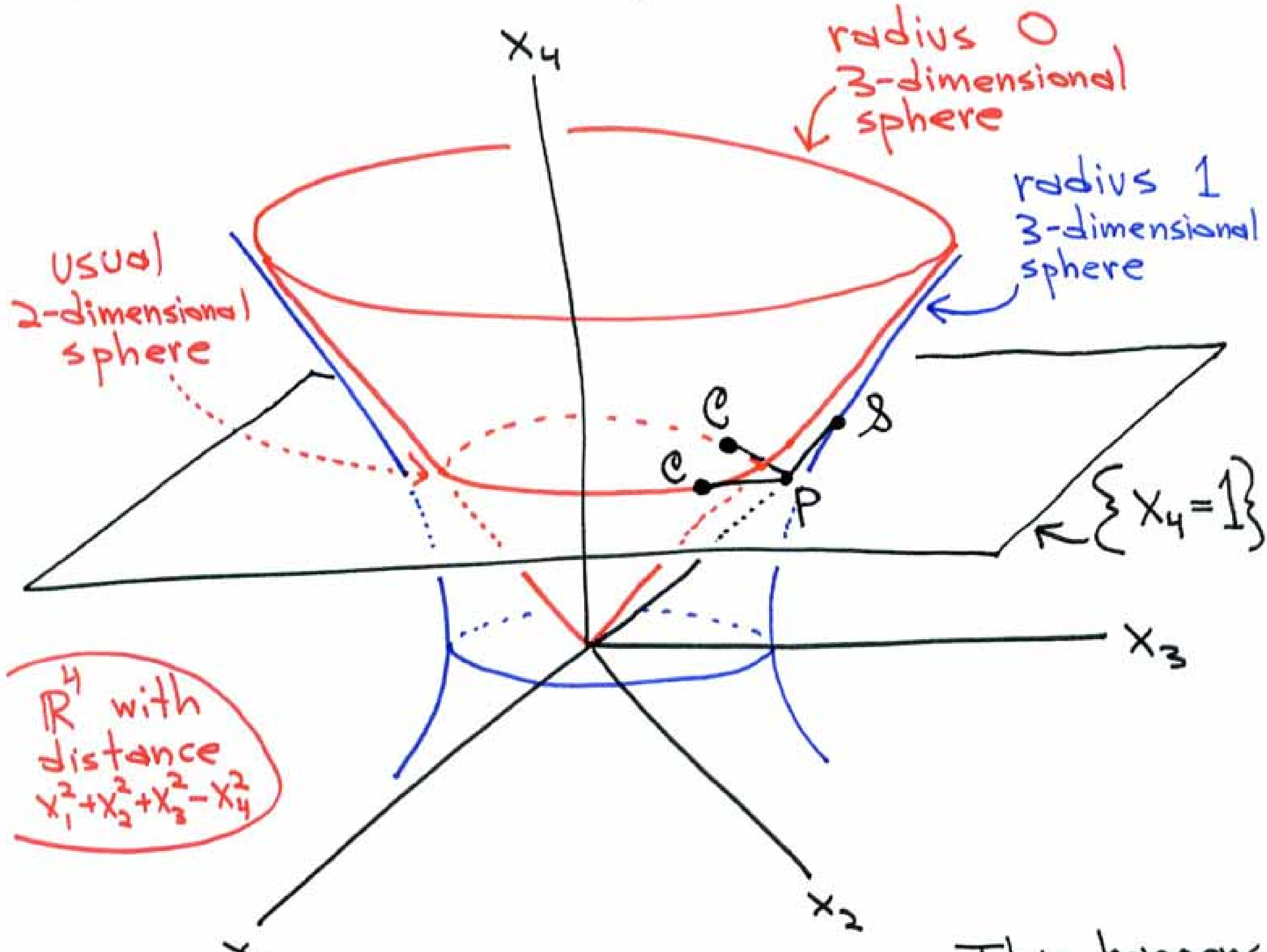
$\|\vec{B}\|^2 = 0$

$\|\vec{C}\|^2 = -1$

Now "spheres", i.e. sets of constant distance from a point, look like:



Now we can relate spheres with points in other spheres:

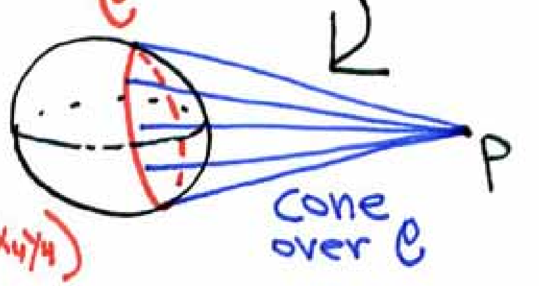


$\mathcal{S}$ is now giving us a 1-dimensional sphere (circle)

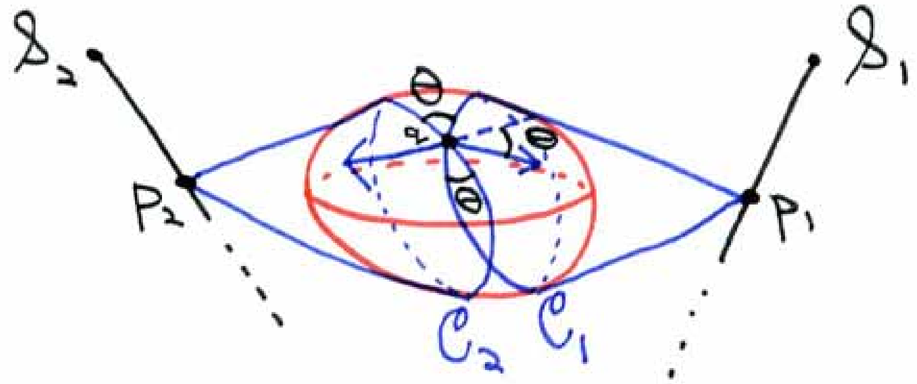
$$\mathcal{C} = \{ \vec{X} \in \mathbb{R}^4 \mid \|\vec{X}\|^2 = 0, \langle \vec{X}, \mathcal{S} \rangle = 0, x_4 = 1 \}.$$

(here $\langle (x_1,x_2,x_3,x_4),(y_1,y_2,y_3,y_4) \rangle = x_1y_1+x_2y_2+x_3y_3-x_4y_4$)

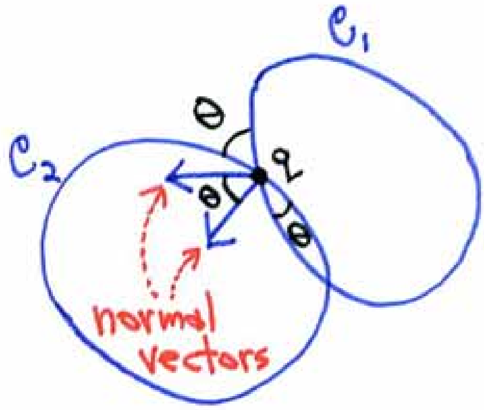
This happens inside $\{x_4=1\}$.



To compute angles between spheres:



planar picture would be:



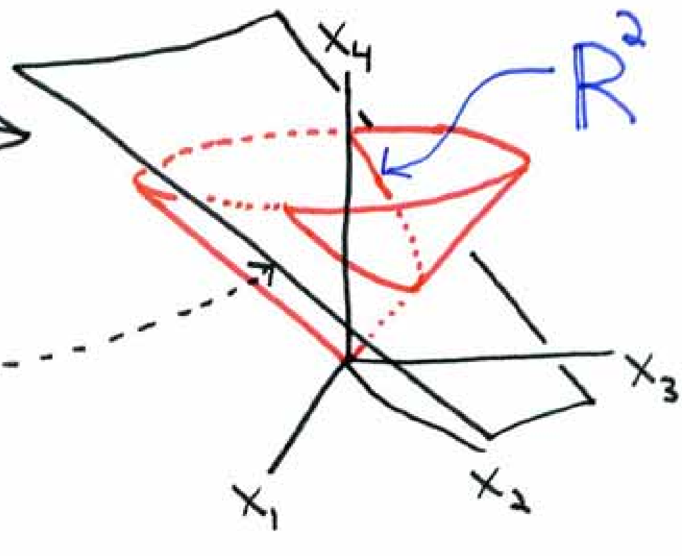
So $\cos\theta = \pm \dfrac{\langle q-p_1, q-p_2\rangle}{\|q-p_1\|\cdot\|q-p_2\|} = \pm \langle \mathcal{S}_1, \mathcal{S}_2 \rangle$,

using

$$\langle q, q\rangle = \|q\|^2 = 0,$$
$$\langle q, p_1\rangle = \langle q, p_2\rangle = 0,$$
$$\mathcal{S}_1 = \frac{p_1}{\|p_1\|}, \quad \mathcal{S}_2 = \frac{p_2}{\|p_2\|}.$$

<u>Theorem</u> The cosine of the angle between $\mathcal{C}_1$ and $\mathcal{C}_2$ is $\pm\langle \mathcal{S}_1, \mathcal{S}_2\rangle$.

The above story can be done with $\mathbb{R}^2$ as well, if we use a different slice.

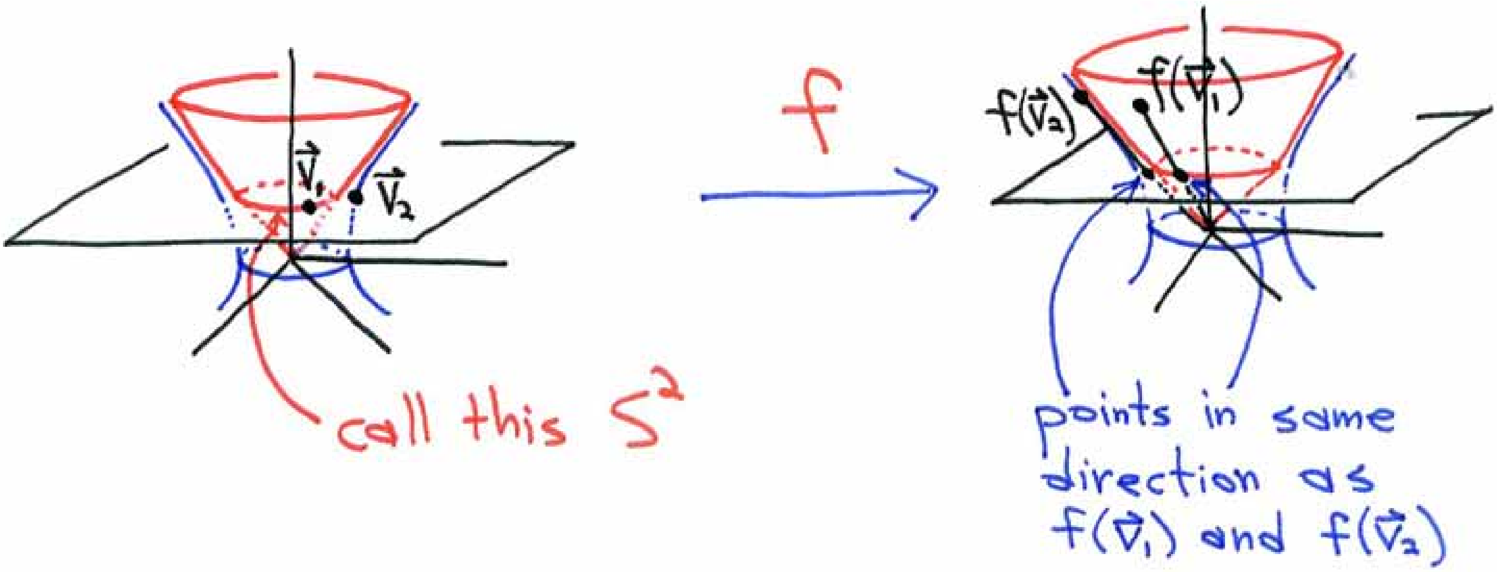


Isometries (等長変換) of $\mathbb{R}^4$ with "distance" $x_1y_1 + x_2y_2 + x_3y_3 - x_4y_4$ are those maps that preserve that distance:

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} \xrightarrow{\ f\ } A \cdot \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix},$$

where A is a 4×4 matrix so that

$$A \cdot \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \cdot A^{t} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$



So f, as a map from S^2 to S^2, does this:

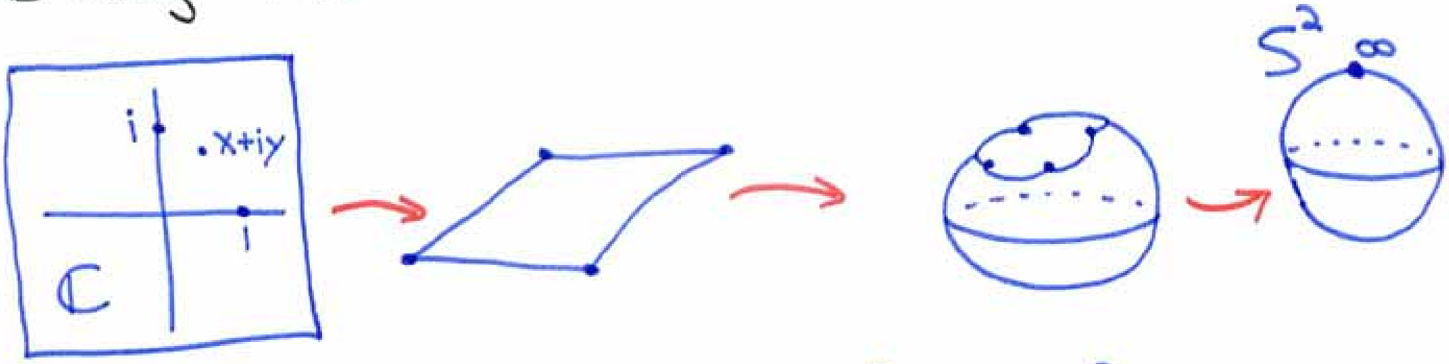
points in $S^2 \longrightarrow$ points in S^2,

spheres (circles) in $S^2 \longrightarrow$ spheres in S^2.

These are what we call

$\boxed{\text{Möbius transformations (メービウス変換)}}$

Doing this:



Then the map $f: S^2 \to S^2$ becomes

$$z \xrightarrow{f} \frac{az+b}{cz+d}$$

($a,b,c,d \in \mathbb{C}$ constant, $ad-bc \neq 0$).

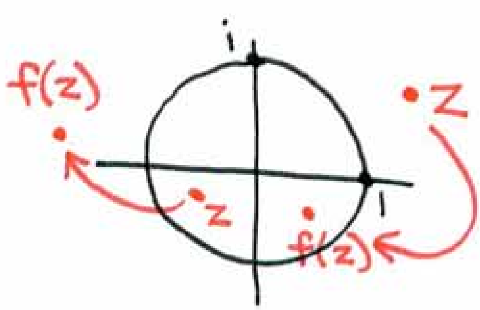
Examples

1) translation (平行移動): $z \xrightarrow{f} z+b$

2) dilation (相似変換): $z \xrightarrow{f} r \cdot z$ ($r \in \mathbb{R}$)

3) rotation (回転): $z \xrightarrow{f} e^{i\theta} \cdot z$ ($\theta \in \mathbb{R}$)

4) inversion (反転): $z \xrightarrow{f} \frac{1}{z}$



5) "square root" of inversion: $z \xrightarrow{f} \frac{z+i}{iz+1}$

(then $f(f(z)) = \frac{1}{z}$)